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Simultaneous Pell Equations $x^2 - my^2 = 1$ and $y^2 - pz^2 = 1$

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ABSTRACT

Pell equation is a special type of Diophantine equations of the form $x^2 - my^2 = 1$, where m is a positive non-square integer. Since m is not a perfect square, then there exist infinitely many integer solutions (x, y) to the Pell equation. This paper will discuss the integral solutions to the simultaneous Pell equations $x^2 - my^2 = 1$ and $y^2 - pz^2 = 1$, where m is square free integer and p is odd prime. The solutions of these simultaneous equations are of the form of $(x, y, z, m) = ({y_n}^2 t \pm 1, y_n, z_n, {y_n}^2 t^2 \pm 2t)$ and $(\frac{y_n^2}{2} t \pm 1, y_n, z_n, \frac{y_n^2}{4} t^2 \pm t)$ for y_n odd and even respectively, where $t \in \mathbb{N}$.

Keywords: Simultaneous Pell equations, Pell equation and parity.

1. Introduction

The fundamental solution, which is denoted by $\varepsilon_1 = x_1 + y_1\sqrt{m}$ is the smallest integer solution for the Pell equation $x^2 - my^2 = 1$. Tekcan (2011) gave the formula of finding the fundamental solution based on the continued fraction expansion of $\sqrt{m}$ for $m = u^2 \pm 1, u^2 \pm 2$, and $u^2 \pm u$, where $u \in \mathbb{N}$. The fundamental solution can be used to generate all other solutions (x_n, y_n) of Pell equation which is given by the following formula

$$x_n + y_n\sqrt{m} = (x_1 + y_1\sqrt{m})^n, \tag{1}$$

where $n \in \mathbb{N}$. Nagell (1964) provided a formula to find the value of x_n and y_n which are of the form

$$\begin{cases} x_n = x_1^n + \sum\limits_{k=1} \binom{n}{2k} x_1^{n-2k} y_1^{2k} m^k, \\ y_n = \sum\limits_{k=1} \binom{2}{2k-1} x_1^{n-2k+1} y_1^{2k-1} m^{k-1}. \end{cases}$$

Anglin (1996) had considered the simultaneous Pell equations $x^2 - Ry^2 = 1$ and $z^2 - Sy^2 = 1$ with $R < S \leq 200$ and all the positive solutions are given by

$$\begin{cases} x = u_r = \frac{A^r + A^{-r}}{2}, \quad & y = v_r = \frac{A^r + A^{-r}}{2\sqrt{R}}, \quad & \text{for } x^2 - Ry^2 = 1, \\ z = w_n = \frac{A'^n + A'^{-n}}{2}, \quad & y = t_n = \frac{A'^n + A'^{-n}}{2\sqrt{S}}, \quad & \text{for } z^2 - Sy^2 = 1, \end{cases}$$

where A and A' is the fundamental solution of $x^2 - Ry^2 = 1$ and $z^2 - Sy^2 = 1$ respectively and $r, n = 0, 1, 2, \ldots$. Anglin conclude that there are no solutions (x, y, z) to the simultaneous Pell equations for both r and n are greater than 2. The only solution for the case of either r or n more than 2 is when $r = 3$ and $n = 1$. There are no solution for $y > 120$.

By considering the method of simultaneous Padé approximation to hypergeometric functions with a gap principle, Bennett (1998) studied the number of solutions to the simultaneous Pell equations $x^2 - az^2 = 1$ and $y^2 - bz^2 = 1$, where a and b are distinct nonzero integers and proved that there exist at most three positive integral solutions (x, y, z). Bennett also managed to prove that the number of integral solutions to the simultaneous Pell equations $x^2 - ay^2 = 1$ and $y^2 - bz^2 = 1$ is at most three.

By assuming $a = 4m(m+1)$ with m is nonzero positive integer, Yuan (2004) proved that the simultaneous Pell equations $x^2 - 4m(m+1)y^2 = 1$ and $y^2 - bz^2 = 1$ has at most one positive integral solution (x, y, z).

Ai et al. (2015) provided the complete solutions to the simultaneous Pell equations in Yuan (2004) for $m = 2$ and b is prime, denoted by p. The solutions of simultaneous equations $x^2 - 24y^2 = 1$ and $y^2 - pz^2 = 1$ are $(x, y, z, p) = (49, 10, 3, 11)$ and $(485, 99, 70, 2)$.

Let m be positive square free integer and p be odd prime. In this paper, we will study the integral solutions for the simultaneous Pell equations

$$x^2 - my^2 = 1 \text{ and } y^2 - pz^2 = 1. \tag{2}$$

In order to find the solutions, we need the following definition.

Definition 1.1. *(Divisibility) An integer a is said to be divisible by an integer $d \neq 0$, denoted as $d|a$ if there exist some integer c such that $a = dc$.*

2. Main Result

The following theorem will give the integral solutions to the simultaneous Pell equations $x^2 - my^2 = 1$ and $y^2 - pz^2 = 1$ based on parity of y_n.

Theorem 2.1. *Let x, y, z, m be positive integers and p odd prime. The integral solutions to the simultaneous Pell equations $x^2 - my^2 = 1$ and $y^2 - pz^2 = 1$ is of the form*

$$(x, y, z, m) = \begin{cases} (\frac{y_n^2}{2}t \pm 1, y_n, z_n, \frac{y_n^2}{4}t^2 \pm t), & \text{if } y_n \text{ is even} \\ (y_n^2 t \pm 1, y_n, z_n, y_n^2 t^2 \pm 2t), & \text{if } y_n \text{ is odd} \end{cases}$$

for $n, t \in \mathbb{N}$.

Proof. From (1), n-th solution is given by $\varepsilon_n = y_n + z_n\sqrt{p}$. We will consider two cases, in which y_n is even or odd.

CASE A: Suppose, y_n is even.
In this case, we will consider another two cases as follows:

Case I: Suppose, m even and x odd integer.

Let $m = 2^\alpha k$ and $x = 2^\gamma s + 1$, where $(2, k) = (2, s) = 1$ and $\alpha, \gamma \geq 1$. By substituting the values of x and m into $x^2 - my_n^2 = 1$, we will obtain

$$2^{2\gamma} s^2 + 2^{\gamma+1} s = 2^\alpha k y_n^2. \tag{3}$$

In order to get the expressions of m, we will consider another three cases, $\alpha = \gamma$, $\alpha < \gamma$ and $\alpha > \gamma$.

(i) The first case if $\alpha = \gamma$:
From (3), we obtain

$$k = \frac{2^\gamma s^2 + 2s}{y_n^2}. \tag{4}$$

We substitute (4) into $m = 2^\alpha k$, we will have

$$m = \frac{2^{2\gamma} s^2 + 2^{\gamma+1} s}{y_n^2}. \tag{5}$$

We consider two possibilities of factorization, which are

$$m = \frac{2^{\gamma+1} s(2^{\gamma-1} s + 1)}{y_n^2}, \tag{6}$$

or

$$m = \frac{s(2^{2\gamma} s + 2^{\gamma+1})}{y_n^2}. \tag{7}$$

From (6) and m is integer, clearly that $y_n^2 | 2^{\gamma+1} s$ or $y_n^2 | (2^{\gamma-1} s + 1)$.

From Definition 1.1, $y_n^2 | 2^{\gamma+1} s$ then, there exists $t \in \mathbb{N}$ such that $2^{\gamma+1} s = y_n^2 t$. Hence,

$$m = \frac{2^{\gamma+1} s(2^{\gamma-1} s + 1)}{y_n^2} = \frac{y_n^2 t(y_n^2 t \cdot 2^{-2} + 1)}{y_n^2} = \frac{y_n^2}{4} t^2 + t. \tag{8}$$

For the value of x, we will have

$$x = 2^\gamma s + 1 = y_n^2 t \cdot 2^{-1} + 1 = \frac{y_n^2}{2} t + 1. \tag{9}$$

Next, we consider for the case $y_n^2 | (2^{\gamma-1} s + 1)$. Since y_n^2 is even, it only divides even number. So, $2^{\gamma-1} s + 1$ is even if $\gamma = 1$. We have $y_n^2 | (s + 1)$, then there exists $t \in \mathbb{N}$ such that $s + 1 = y_n^2 t$.

Thus,

$$m = \frac{2^2 s(s + 1)}{y_n^2} = \frac{4(y_n^2 t - 1) y_n^2 t}{y_n^2} = 4(y_n^2 t^2 - t). \tag{10}$$

For the value of x, we will have

$$x = 2s + 1 = 2(y_n^2 t - 1) + 1 = 2y_n^2 t - 1. \tag{11}$$

Let $t = \frac{t_1}{4}$ for some $t_1 \in \mathbb{N}$, equations (10) and (11) become

$$m = \frac{y_n^2}{4} t_1^2 - t_1 \text{ and } x = \frac{y_n^2}{2} t_1 - 1$$

respectively.

From (7), since s is odd, then $y_n^2 \nmid s$. Clearly that $y_n^2 | (2^{2\gamma} s + 2^{\gamma+1})$, then there exists $t \in \mathbb{N}$ such that

$$s = \frac{y_n^2 t - 2^{\gamma+1}}{2^{2\gamma}}. \tag{12}$$

By substituting (12) into (7), we obtain

$$m = \frac{y_n^2}{2^{2\gamma}} t^2 - 2^{1-\gamma} t. \tag{13}$$

Then, substitute (12) into $x = 2^\gamma s + 1$, we will have

$$x = \frac{y_n^2}{2^\gamma} t - 1 \tag{14}$$

with $\gamma \geq 1$.
Let $t = \frac{2^{\gamma+1} t_2}{4}$ for some $t_2 \in \mathbb{N}$, equations (13) and (14) become

$$m = \frac{y_n^2}{4} t_2^2 - t_2 \text{ and } x = \frac{y_n^2}{2} t_2 - 1$$

respectively.

(ii) Now, we consider the second case if $\alpha < \gamma$, as follows:
From (3), we obtain

$$k = \frac{2^{2\gamma-\alpha} s^2 + 2^{\gamma-\alpha+1} s}{y_n^2}. \tag{15}$$

Substitute (15) into $m = 2^\alpha k$ and we will obtain (5). Next, we will consider (6) and (7).

The factorization of (6) with m integer, will have $\gamma > 1$ and clearly that $y_n^2 | 2^{\gamma+1} s$ since $y_n^2 \nmid (2^{\gamma-1} s + 1)$.

Then, there exists $t \in \mathbb{N}$ such that

$$2^{\gamma+1}s = y_n^2 t. \tag{16}$$

By substituting (16) into (6) and $x = 2^{\gamma}s + 1$, then we will obtain (8) and (9) respectively.

Now, we consider (7) with m is integer. Since $y_n^2 \nmid s$, we will have (12). The solutions for m and x are of the form (13) and (14) with $\gamma > 1$ since our case is $\gamma > \alpha \geq 1$.

(iii) Lastly, we consider the case, if $\alpha > \gamma$:
From (3), we obtain

$$2^{\gamma-1}s^2 + s = 2^{\alpha-\gamma-1}ky_n^2. \tag{17}$$

Right-hand side of (17) is always even, then the left-hand side will be even when $\gamma = 1$. Then we will have

$$k = \frac{s^2 + s}{2^{\alpha-2}y_n^2}. \tag{18}$$

We substitute (18) into $m = 2^{\alpha}k$, and obtain

$$m = \frac{4s(s+1)}{y_n^2}. \tag{19}$$

Since m is integer, then clearly that $y_n^2 | 4s$ or $y_n^2 | (s+1)$.

From Definition 1.1, $y_n^2 | 4s$ implies that there exists $t \in \mathbb{N}$ such that $4s = y_n^2 t$. Hence we will obtain (8) and (9) from (19) and $x = 2^{\gamma}s + 1$ respectively.

Next, we consider for the case $y_n^2 | (s+1)$. From Definition 1.1, $y_n^2 | (s+1)$ implies that there exists $t \in \mathbb{N}$ such that $s + 1 = y_n^2 t$. Hence, from (19) and $x = 2^{\gamma}s + 1$ we obtain (10) and (11) respectively.

Now, we consider the second case.

Case II: Suppose, m and x are odd integers.

Let $m = 2^{\alpha}k + 1$ and $x = 2^{\gamma}s + 1$, where $(2, k) = (2, s) = 1$ and $\alpha, \gamma \geq 1$. Substituting the values x and m into $x^2 - my_n^2 = 1$, we obtain

$$2^{2\gamma}s^2 + 2^{\gamma+1}s - y_n^2 = 2^{\alpha}ky_n^2. \tag{20}$$

We will consider the same cases as Case I, that is $\alpha = \gamma, \alpha > \gamma$ and $\alpha < \gamma$.

(i) For the case $\alpha = \gamma$, we have as follows:
From (20), we have

$$k = \frac{2^{2\gamma}s^2 + 2^{\gamma+1}s - y_n^2}{2^{\gamma}y_n^2}. \tag{21}$$

Substituting (21) into $m = 2^{\alpha}k + 1$, we obtain (5). Then we consider the factorizations of (6) and (7).

By using the same argument as in Case I, we obtain the solutions as (8), (9), (10), (11), (13) and (14).

(ii) For the case if $\alpha < \gamma$ and $\alpha > \gamma$, we have as follows :
By simplifying (20), we have

$$k = \frac{2^{2\gamma}s^2 + 2^{\gamma+1}s - y_n^2}{2^{\alpha}y_n^2}. \tag{22}$$

Substitute (22) into $m = 2^{\alpha}k + 1$, we have (5), (6) and (7). By using the same argument as in Case I, we obtain (8), (9), (13), and (14) for case $\alpha < \gamma$ and for case $\alpha > \gamma$, we obtain (8), (9), (10), (11), (13) and (14).

CASE B: Suppose, y_n is odd.
We will consider two cases. That is:
Case I: m is even, x is odd
Case II: m is odd, x is even

Case I: Let $m = 2^{\alpha}k$ and $x = 2^{\gamma}s + 1$, where $(2, k) = (2, s) = 1$ and $\alpha, \gamma \geq 1$. Substituting the values of m and x into $x^2 - my_n^2 = 1$, we obtain (3). From this equation, we consider another two cases. That is, $\alpha \leq \gamma$ or $\alpha > \gamma$.

(i) Now, we consider the first case. That is $\alpha \leq \gamma$.
From (3), we obtain

$$2^{2\gamma-\alpha}s^2 + 2^{\gamma-\alpha+1}s = y_n^2 k. \tag{23}$$

From (23), it is contradiction since y_n is odd yield the right-hand side is always odd but the left-hand side is always even.

(ii) For the second case which is $\alpha > \gamma$, we have the following:
From (3), we obtain (17). On the right-hand side of (17) is even when $\alpha - \gamma - 1 > 0$ and left-hand side is even when $\gamma = 1$. Then we will obtain the values of k and m as (18) and (19).

We will consider two possibilities of factorization, which are (19) or

$$m = \frac{2s(2s+2)}{y_n^2}. \tag{24}$$

From (24) and m integer, clearly that $y_n^2 | 2s$ or $y_n^2 | (2s+2)$.
From Definition 1.1, $y_n^2 | 2s$ implies that there exists $t \in \mathbb{N}$ such that $2s = y_n^2 t$. Thus, we have

$$m = y_n^2 t^2 + 2t \ \text{ and } \ x = y_n^2 t + 1. \tag{25}$$

Now, we consider for the case $y_n^2 | (2s+2)$, then exists $t \in \mathbb{N}$ such that $2s + 2 = y_n^2 t$. We obtain

$$m = y_n^2 t^2 - 2t \ \text{ and } \ x = y_n^2 t - 1. \tag{26}$$

Next, we focus on (19). It is clear that $y_n^2 | 4s$ or $y_n^2 | (s+1)$. From Definition 1.1, $y_n^2 | 4s$ implies that there exists $t \in \mathbb{N}$ such that $4s = y_n^2 t$. Then, we will obtain (8) and (9). Let $t = 2t_3$ for $t_3 \in \mathbb{N}$, (8) and (9) become

$$m = y_n^2 t_3^2 + 2t_3 \ \text{ and } \ x = y_n^2 t_3 + 1$$

respectively which has similar pattern as (25).

Now, suppose $y_n^2 | (s+1)$ which implies that there exists $t \in \mathbb{N}$ such that $s + 1 = y_n^2 t$. For this case, we obtain (10) and (11). Let $t = \frac{t_4}{2}$ for some $t_4 \in \mathbb{N}$, (10) and (11) become

$$m = y_n^2 t_4^2 - 2t_4 \ \text{ and } \ x = y_n^2 t_4 - 1$$

respectively which has similar pattern as (26).

From (17), the right-hand side is odd when $\alpha = \gamma + 1$ and the left-hand side is odd when $\gamma > 1$. Then, we will obtain

$$k = \frac{2^{\gamma-1}s^2 + s}{y_n^2}. \tag{27}$$

Substituting (27) into $m = 2^\alpha k$, we obtain m of the form (5). Next, we consider two possibilities as in (6) or

$$m = \frac{2^\gamma s(2^\gamma s + 2)}{y_n^2}. \tag{28}$$

By Definition 1.1, we yield the solutions of the form (25) and (26) for (28) and $x = 2^\gamma s + 1$.

For equation (6) and $x = 2^\gamma s + 1$, we obtain the similar results as (25) and (26).

Consider Case II: Let $m = 2^\alpha k + 1$ and $x = 2^\gamma s$, where $(2, k) = (2, s) = 1$ and $\alpha, \gamma \geq 1$.

We substitute the values of m and x into $x^2 - y_n^2 m = 1$, we obtain

$$2^{2\gamma}s^2 - (1 + y_n^2) = 2^\alpha k \cdot y_n^2. \tag{29}$$

For this case, we consider three cases as follows:

(i) The first case if $\alpha = \gamma$.
From (29), we have

$$2^\gamma s^2 - 2^{-\gamma}(1 + y_n^2) = y_n^2 k. \tag{30}$$

Right-hand side of (30) is always odd and left-hand side will be odd if $2^{-\gamma}(1 + y_n^2)$ is odd.

Let $y_n = 2^\beta j + 1$, with $(2, j) = 1$ and $\beta \geq 1$. Then,

$$y_n^2 + 1 = 2(2^{2\beta-1}j^2 + 2^\beta j + 1). \tag{31}$$

Substituting (31) into (30), we get

$$2^\gamma s^2 - 2^{1-\gamma}(2^{2\beta-1}j^2 + 2^\beta j + 1) = y_n^2 k. \tag{32}$$

The left-hand side of (32) is odd when $\gamma = 1$. Then,

$$k = \frac{4s^2 - y_n^2 - 1}{2y_n^2}. \tag{33}$$

Substituting (33) into $m = 2^\alpha k + 1$, we have $m = \frac{(2s-1)(2s+1)}{y_n^2}$ and obtain (25) and (26).

(ii) For the second case, we consider if $\alpha < \gamma$.
Substituting (31) into (29), we have

$$2^{2\gamma-\alpha}s^2 - 2^{1-\alpha}(2^{2\beta-1}j^2 + 2^\beta j + 1) = y_n^2 k. \tag{34}$$

Left-hand side of (34) will be odd if $\alpha = 1$. Then,

$$k = \frac{2^{2\gamma}s^2 - y_n^2 - 1}{2y_n^2}. \tag{35}$$

Substituting (35) into $m = 2^\alpha k + 1$, we have $m = \frac{(2^\gamma s-1)(2^\gamma s+1)}{y_n^2}$ and obtain (25) and (26).

(iii) Lastly, if $\alpha > \gamma$.
Substituting (31) into (29), we obtain

$$2^{2\gamma-1}s^2 - 2^{2\beta-1}j^2 - 2^\beta j - 1 = 2^{\alpha-1}ky_n^2. \tag{36}$$

It is contradict since right-hand side of (36) is even but the left-hand side is always odd.

$\square$

3. Conclusion

By considering the parity of y, m and x, the solutions to the simultaneous Pell equations $x^2 - my^2 = 1$ and $y^2 - pz^2 = 1$, where m is square free integer and p is odd prime is of the form

$$(x, y, z, m) = \begin{cases} (\frac{y_n^2}{2}t \pm 1, y_n, z_n, \frac{y_n^2}{4}t^2 \pm t), & \text{if } y_n \text{ is even} \\ (y_n^2 t \pm 1, y_n, z_n, y_n^2 t^2 \pm 2t), & \text{if } y_n \text{ is odd} \end{cases}$$

for $n, t \in \mathbb{N}$.

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